SUMMATION NOTATION

1. We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned}
\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\
&= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\
&= 0.13131313
\end{aligned}$$

2. We replace every occurrence of n with the values 0 through 4.

$$\begin{aligned}
\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} + \frac{4!}{2} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{2 \cdot 1}{2} + \frac{3 \cdot 2 \cdot 1}{2} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2} \\
&= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\
&= 17
\end{aligned}$$

3. We replace the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}
\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\
&\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}
\end{aligned}$$

4. The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. Hence, $a_n = 1 + (n-1)2 = 2n - 1$, $n \geq 1$.

   At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + \ldots + 117 = \sum_{n=1}^{59} (2n - 1)$$

5. We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \ldots + \frac{1}{117}$$

   The numerators, 1, -1, etc. can be described by the geometric sequence $(-1)^{n-1}$ for $n \geq 1$, while the denominators are given by n, $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

6. One formula for the n^{th} term is $a_n = (0.9)(0.1)^{n-1}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation.

   To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need to go to the nth term. Hence, n is the upper limit of summation. Since we are using n as the upper limit of summation, we need to change our counter to a different letter, say k, so we add $a_k = (0.9)(0.1)^{k-1}$ from $k = 1$ to $k = n$:

$$0.9 + 0.09 + 0.009 + \ldots 0.\underbrace{0 \cdots 0}_{n-1 \text{ zeros}}9 \quad = \quad \sum_{k=1}^{n} (0.9)(0.1)^{k-1}$$

7. Using the Sum and Difference Property along with the Distributive Property we get:

$$\sum_{n=2}^{50} (a_n - 3b_n) = \sum_{n=2}^{50} a_n - \sum_{n=2}^{50} 3b_n = \sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n$$

   Hence, $\displaystyle\sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n = 17$. If $\displaystyle\sum_{n=2}^{50} a_n = 10$, then $\displaystyle 10 - 3\sum_{n=2}^{50} b_n = 17$ so $\displaystyle\sum_{n=2}^{50} b_n = -\frac{7}{3}$.

8. There are at least two ways to approach this problem. By definition, $\displaystyle\sum_{n=1}^{21} a_n = a_1 + a_2 + \ldots + a_{21}$. That is, we add up the first 21 terms of the sequence a_n. Similarly, $\displaystyle\sum_{n=1}^{20} a_n = a_1 + a_2 + \ldots + a_{20}$ means we add up the first 20 terms of the sequence. Hence, $\displaystyle a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$.

   Alternatively, we can use the Additive Index Property:

$$\sum_{n=1}^{21} a_n = \sum_{n=1}^{20} a_n + \sum_{n=21}^{21} a_n = \sum_{n=1}^{20} a_n + a_{21},$$

   which gives $\displaystyle a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$ as well.

9. To re-index $\displaystyle\sum_{n=2}^{437} n(n-1)x^{n-2}$ so n starts at 0, we follow the formula above with $r = -2$:

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{n=2+(-2)}^{437+(-2)} (n-(-2))(n-(-2)-1)x^{n-(-2)-2} = \sum_{n=0}^{435} (n+2)(n+1)x^n.$$

   We leave it to the reader to check by writing out the first few, and last few, terms.

   Alternatively, to better see *why* the re-indexing works in this way, we can introduce a new counter, k. We want this new counter to start at $k = 0$ whereas the current counter starts at $n = 2$, so we want $k = n-2$. When $n = 2$, $k = 0$, as required, and when $n = 437$, $k = 435$.

   Moreover, $n = k + 2$, so substituting this into the sum, we get

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{k=0}^{435} (k+2)((k+2)-1)x^{(k+2)-2} = \sum_{k=0}^{435} (k+2)(k+1)x^k,$$

   which is the same sum we had before, just with a different dummy variable.

10. Recognizing the terms of $1+3+5+\ldots+117$ as 1, 3, 5, and so on, we see we have an arithmetic sequence with $a=1$ and $d=2$. We get a formula for the terms $a_n = 1+2(n-1) = 2n-1$ for $n \geq 1$. In order to use the formula given above, we need to determine the number of terms being added, n. Setting $2n-1=117$, we find $n=59$. We get our final answer: $1+3+5+\ldots+117 = 59\left(\frac{1+117}{2}\right) = 3481$.

11. Applying the adage 'when in doubt, write it out,' we have $\sum_{k=1}^{n} k = 1+2+3+4+\ldots+n$. We see the terms here form an arithmetic sequence with $a=d=1$. Moreover, we are adding exactly n terms, we get $\displaystyle\sum_{k=1}^{n} k = 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$.

12. Writing out the amount of money gives the sum: $0.01+0.02+0.04+\ldots$. The terms form a geometric sequence with $a=0.01$ and $r=2$. Hence, we are adding $a_n = (0.01)\left(2^{n-1}\right)$ for $n \geq 1$. We get:

    - After one week: $\displaystyle\sum_{n=1}^{7}(0.01)\left(2^{n-1}\right) = \$1.27$
    - After two weeks: $\displaystyle\sum_{n=1}^{14}(0.01)\left(2^{n-1}\right) = \$163.83$
    - After three weeks: $\displaystyle\sum_{n=1}^{21}(0.01)\left(2^{n-1}\right) = \$20,971.51$
    - After one month: $\displaystyle\sum_{n=1}^{28}(0.01)\left(2^{n-1}\right) = \$2,684,354.55$

13. We have $r=0.06$ and $n=12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P=50$ and $t=30$,

$$A = \frac{50\left((1+0.005)^{(12)(30)}-1\right)}{0.005} \approx 50225.75$$

    Our final answer is \$50,225.75.

14. To find how long it will take for the annuity to grow to \$100,000, we set $A=100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{aligned}
100000 &= \frac{50\left((1+0.005)^{12t}-1\right)}{0.005} \\
10 &= (1.005)^{12t}-1 \\
(1.005)^{12t} &= 11 \\
\ln\left((1.005)^{12t}\right) &= \ln(11) \\
12t\ln(1.005) &= \ln(11) \\
t &= \tfrac{\ln(11)}{12\ln(1.005)} \approx 40.06
\end{aligned}$$

    This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering.